Multiplicative rule of Schubert classes

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Abstract

Let G be a compact connected Lie group and H, the centralizer of a one-parameter subgroup in G. Combining the ideas of Bott-Samelson resolutions of Schubert varieties and the enumerative formula on a twisted product of 2 spheres obtained in [Du$_2$], we obtain an explicit formula for multiplying Schubert classes in the flag manifold G/H.

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1 Introduction

Let G be a compact connected Lie group and H, the centralizer of a one-parameter subgroup in G. The Weyl of G (resp. of H) is denoted by W (resp. W'). The set W/W' of left cosets of W' in W can be identified with the subset of W:

$$\overline{W} = \{w \in W \mid l(w_1) \geq l(w) \text{ for all } w_1 \in wW'\},$$

where $l : W \to \mathbb{Z}$ is the length function relative to a fixed maximal torus T in G.

It is known from Bruhat-Chevalley that the flag manifold $G/H = \{gH \mid g \in G\}$ admits a canonical decomposition into cells, indexed by elements of $\overline{W}$,

$$G/H = \underset{w\in\overline{W}}{\cup} X_w(H), \quad \dim X_w = 2l(w),$$

with each cell $X_w(H)$ the closure of an algebraic affine space, known as a *Schubert variety* in G/H [BGG]. Since only even dimensional cells are involved, the set of fundamental classes $[X_w(H)] \in H_{2l(w)}(G/H)$, $w \in \overline{W}$, form an additive basis of the homology $H_*(G/H)$. The cocycle class $P_w(H) \in H^{2l(w)}(G/H)$, $w \in \overline{W}$, defined by the Kronecker pairing as $\langle P_w(H), [X_u(H)]\rangle = \delta_{w,u}$, $w, u \in \overline{W}$, is called the *Schubert class corresponding to* w. Clearly one has

Basis Theorem. *The set of Schubert classes* $\{P_w(H) \mid w \in \overline{W}\}$ *constitutes an additive basis for the cohomology* $H^*(G/H)$.

One immediate consequence is that the product of two arbitrary Schubert classes can be expressed in terms of Schubert classes. Precisely, given $u, v \in \overline{W}$, one has the expression

$$P_u(H) \cdot P_v(H) = \sum_{l(w)=l(u)+l(v), w\in\overline{W}} a_{u,v}^w P_w(H),\ a_{u,v}^w \in \mathbb{Z}$$

in $H^*(G/H)$. Since the Chow ring $A^*(G/H)$ is canonically isomorphic to the integral cohomology $H^*(G/H)$, the following Problem is of fundamental importance in the intersection theory of G/H.

Problem. *Find the number $a_{u,v}^w$ for given $w, u, v \in \overline{W}$, $l(w) = l(u) + l(v)$.*

If G is the unitary group $U(n)$ of rank n and $H = U(k) \times U(n-k)$, the flag manifold G/H is the Grassmannian $G_{n,k}$ of k-planes through the origin in $\mathbb{C}^n$. In this case, a combinatorial description for $a_{u,v}^w$ is given by the *Littlewood-Richardson rule*, one cornerstone of the Schubert calculus for $G_{n,k}$ [S]. It was first stated by Littlewood and Richardson in 1934 [LR]. Complete proofs appeared only in the 1970s (see "Note and references" in [M, p.148]).

Another special case is when $H = T$ (a maximal torus in G) and if either $l(u) = 1$ or $l(v) = 1$. The number $a_{u,v}^w$ is seen as certain Cartan number of G from the *Chevalley formula*. Chevalley announced the formula at the end of his address at the 1958 ICM in Edinburgh [Ch$_1$], while a proof was given in his famous manuscript [Ch$_2$]. Although [Ch$_2$] remained unpublished until 1994, this formula became part of the official literature after the publications of [BGG] by Bernstein et al in 1973, and [De$_2$] by Demazure in 1974, where both authors verified it using different methods (cf. introduction to [Ch$_2$] by Borel).

In recent years, inspired by theory of Schubert polynomials of Lascoux and Schützenberger [LS], many achievements have been made in generalizing the classical *Pieri formula*, which handles the problem for the case where G is a matrix group and where one of P_u and P_v is a *special Schubert class*. (See [FP, Section 9.10] for more recent progresses and relevant references).

While the problem in its natural generality remains unsolved[1], the further problem of determining the multiplicative rule of Schubert classes in the quantum cohomology of G/H has appeared on the agenda, where the analogue of the coefficients $a_{u,v}^w$ are known as *Gromov-Witten numbers* (cf. [FP, p.134], [CF]).

It was announced in [Du$_2$] that, combining the ideas of Bott-Samelson resolutions of Schubert varieties and the enumerative formula on a twisted products of 2-spheres obtained in [Du$_2$], it is possible to find a unified formula that expresses $a_{u,v}^w$ in terms of certain Cartan numbers of G. This paper is devoted to complete this project.

[1] We quote from Fulton and Pragacz [FP]: *there is no analogue of the Littlewood Richardson rule for explicitly multiplying Schubert classes in a flag manifold*; from Sottile [S]: *the analog of the Littlewood Richardson rule is not known for most other flag variety G/P.*

For the cases of matrix groups, the theory of Schubert polynomials was developed to make explicit computation with Schubert classes possible (cf. introduction to [BH]).

2 Main result

A few notations will be needed in presenting our result. Throughout this paper G is a compact connected Lie group with a fixed maximal torus T. We set $n = \dim T$.

2.1. Geometry of Cartan subalgebra. Equip the Lie algebra $L(G)$ of G with an inner product $(,)$ so that the adjoint representation acts as isometries of $L(G)$. The *Cartan subalgebra* of G is the Euclidean subspace $L(T)$ of $L(G)$.

The restriction of the exponential map $\exp : L(G) \to G$ to $L(T)$ defines a set $D(G)$ of $m = \frac{1}{2}(\dim G - n)$ hyperplanes in $L(T)$, i.e. the set of *singular hyperplanes* through the origin in $L(T)$. These planes divide $L(T)$ into finitely many convex cones, called the *Weyl chambers* of G. The reflections σ of $L(T)$ in the these planes generate the Weyl group W of G. Let $\Phi \subset L(T)$ be the *root system associated to* W ([Hu]). Recall that if $\beta, \beta' \in \Phi$, *the Cartan number* $\beta \circ \beta' = 2(\beta, \beta')/(\beta', \beta')$ is an integer (only $0, \pm 1, \pm 2, \pm 3$ can occur).

Fix, once and for all, a regular point $\alpha \in L(T) \setminus \underset{L \in D(G)}{\cup} L$, and let Φ^+ (resp. Δ) be the set of positive roots (resp. simple roots) relative to α. The set $D(G)$ can now be indexed by Φ^+ as $\{L_\beta \mid \beta \in \Phi^+\}$, where L_β is the singular plane corresponding to the root β. For a $\beta \in \Phi^+$, write $\sigma_\beta \in W$ for the reflection of $L(T)$ in L_β. If $\beta \in \Delta$ we call σ_β a *simple reflection*.

It is known that the set of simple reflections $\{\sigma_\beta \mid \beta \in \Delta\}$ generates W. That is, any $w \in W$ admits a factorization of the form

(2.1) $$w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k},\ \beta_i \in \Delta.$$

Definition 1. The *length* $l(w)$ of an $w \in W$ is the least number of factors in all decompositions of w in the form (2.1). The decomposition (2.1) is said *reduced* if $k = l(w)$.

If (2.1) is a reduced decomposition, the $k \times k$ (strictly upper triangular) matrix $A_w = (a_{i,j})$ with

$$a_{i,j} = \left\{ \begin{array}{c} 0 \text{ if } i \geq j; \\ -\beta_i \circ \beta_j \text{ if } i < j \end{array} \right.$$

is called *the Cartan matrix of* w *associated to the decomposition* (2.1).

Example 1. By resorting to the geometry of the Cartan subalgebra $L(T)$ there is geometric method to find a reduced decomposition (hence a Cartan matrix) of an $w \in W$.

Picture W as the W-orbit $\{w(\alpha) \in L(T) \mid w \in W\}$ of the regular point α. Given an $w \in W$ let C_w be a straight line segment in $L(T)$ from the Weyl chamber containing α to $w(\alpha)$ that crosses the planes in $D(G)$ one at a time. Assume that they are met in the order $L_{\alpha_1}, \cdots, L_{\alpha_k}$, $\alpha_i \in \Phi^+$. We have $l(w) = k$ and $w = \sigma_{\alpha_k} \circ \cdots \circ \sigma_{\alpha_1}$ (cf. [Du_2] or [Han]). Set

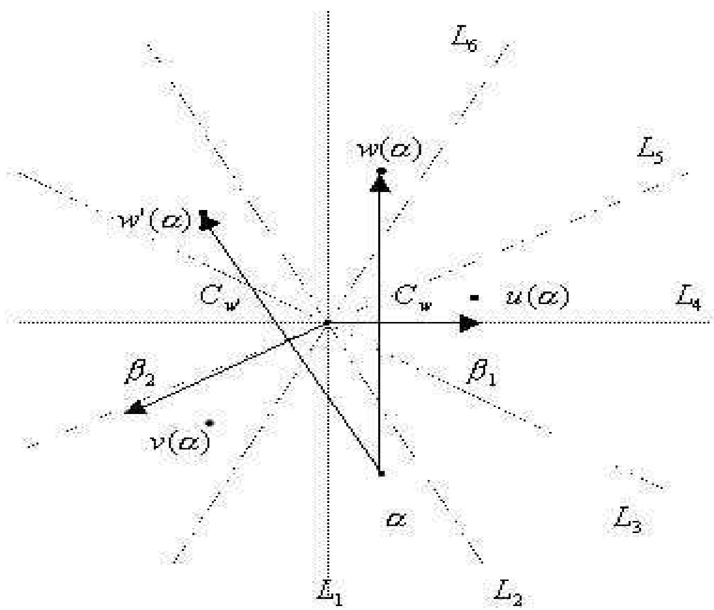
$$\beta_1 = \alpha_1, \quad \beta_2 = \sigma_{\alpha_1}(\alpha_2), \cdots, \quad \beta_k = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{k-1}}(\alpha_k).$$

Then $\beta_i \in \Delta$. Moreover, from

$$\sigma_{\beta_i} = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{i-1}} \circ \sigma_{\alpha_i} \circ \sigma_{\alpha_{i-1}} \circ \cdots \circ \sigma_{\alpha_1},$$

one verifies easily that $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$. This decomposition is reduced because of $\beta_i \in \Delta$ and $l(w) = k$.

As an example of the method consider the case of G_2, the exceptional Lie group of rank 2. With $\dim L(T) = 2$, the singular lines, denoted by L_i, $i \leq 6$, are depicted in the figure. Taking a regular point $\alpha \in L(T)$ as marked, the set of simply roots is $\Delta = \{\beta_1, \beta_2\}$. Let W be the Weyl group of G_2.



For the elements $w, w\prime \in W$ specified by the vectors $w(\alpha), w\prime(\alpha) \in L(T)$ in the figure, we get from the segments C_w and $C_{w'}$ the reduced decompositions

$$w = \sigma_{\beta_2} \circ \sigma_{\beta_1} \circ \sigma_{\beta_2} \circ \sigma_{\beta_1} \circ \sigma_{\beta_2};$$

$$w\prime = \sigma_{\beta_1} \circ \sigma_{\beta_2} \circ \sigma_{\beta_1} \circ \sigma_{\beta_2} \circ \sigma_{\beta_1}.$$

From the Cartan matrix of G_2 ([Hu, p.59])

$$\begin{pmatrix} \beta_1 \circ \beta_1 & \beta_1 \circ \beta_2 \\ \beta_2 \circ \beta_1 & \beta_2 \circ \beta_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

we read that the Cartan matrices of w and w' associated to the decompositions are respectively

$$A_w = \begin{pmatrix} 0 & 3 & -2 & 3 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A_{w'} = \begin{pmatrix} 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 3 & -2 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.2. The triangular operators. Let $\mathbb{Z}[x_1,\cdots,x_k]=\oplus_{r\geq 0}\mathbb{Z}[x_1,\cdots,x_k]^{(r)}$ be the ring of integral polynomials in $x_1,\cdots,x_k$, graded by $\mid x_i\mid=1$.

Definition 2. Given an $k\times k$ strictly upper triangular integer matrix $A=(a_{i,j})$ the *triangular operator* associated to A is the homomorphism $T_A:\mathbb{Z}[x_1,\cdots,x_k]^{(k)}\to\mathbb{Z}$ defined recursively by the following *elimination laws*.

1) if $h\in\mathbb{Z}[x_1,\cdots,x_{k-1}]^{(k)}$, then $T_A(h)=0$;

2) if $k=1$ (consequently $A=(0)$), then $T_A(x_1)=1$;

3) if $h\in\mathbb{Z}[x_1,\cdots,x_{k-1}]^{(k-r)}$ with $r\geq 1$, then

$$T_A(hx_k^r)=T_{A'}(h(a_{1,k}x_1+\cdots+a_{k-1,k}x_{k-1})^{r-1}),$$

where A' is the ($(k-1)\times(k-1)$ strictly upper triangular) matrix obtained from A by deleting the k^{th} column and the k^{th} row.

By additivity, T_A is defined for every $f\in\mathbb{Z}[x_1,\cdots,x_k]^{(k)}$ using the unique expansion $f=\Sigma h_r x_k^r$ with $h_r\in\mathbb{Z}[x_1,\cdots,x_{k-1}]^{(k-r)}$.

Example 2. Definition 2 gives an effective algorithm to evaluate T_A.

For $k=2$ and $A_1=\begin{pmatrix}0&a\\0&0\end{pmatrix}$, then $T_{A_1}:\mathbb{Z}[x_1,x_2]^{(2)}\to\mathbb{Z}$ is given by

$$T_{A_1}(x_1^2)=0,$$
$$T_{A_1}(x_1x_2)=T_{A_1'}(x_1)=1 \text{ and}$$
$$T_{A_1}(x_2^2)=T_{A_1'}(ax_1)=a.$$

For $k=3$ and $A_2=\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}$, then $A_2'=A_1$ and $T_{A_2}:\mathbb{Z}[x_1,x_2,x_3]^{(3)}\to\mathbb{Z}$ is given by

$$T_{A_2}(x_1^{r_1}x_2^{r_2}x_3^{r_3})=\left\{\begin{array}{l}0, \text{ if } r_3=0 \text{ and}\\ T_{A_1}(x_1^{r_1}x_2^{r_2}(bx_1+cx_2)^{r_3-1}),\ \text{if } r_3\geq 1,\end{array}\right.$$

where $r_1+r_2+r_3=3$, and where T_{A_1} is calculated in the above.

It is straightforward from Definition 2 that

Corollary 1. *We have* $T_A(x_1\cdots x_k)=1$ *and*

$$T_A(x_1^{r_1}\cdots x_k^{r_k})=0$$

whenever $r_1+\cdots+r_i>i$ *for some* $1\leq i<k$.

The operator T_A can also be given by explicit formula. For a sequence $(r_1,\cdots,r_k)$ of k non-negative integers with $\Sigma r_i=k$, let $C(r_1,\cdots,r_k)$ be the set of all strictly upper triangular $k\times k$ matrices $C=(c_{i,j})$ with non-negative integer entries satisfying

$$\underset{s}{\Sigma}c_{s,i}=r_i-1+\underset{j}{\Sigma}c_{i,j},\ 1\leq i\leq k.$$

Definition 2, together with induction on k, yields

Corollary 2. $T_A(x_1^{r_1}\cdots x_k^{r_k})=\underset{(c_{i,j})\in C(r_1,\cdots,r_k)}{\Sigma}\underset{j}{\Pi}(\underset{i}{\Sigma}c_{i,j})!\underset{i,j}{\Pi}\frac{a_{i,j}^{c_{i,j}}}{c_{i,j}!}.$

2.3. The formula. It is well known that simply connected semi-simple Lie groups are classified by their Cartan matrices [Hu, p.55]. So, conceivably, any geometric invariant associated to G/H can be reduced in principle to Cartan numbers of G (entries in the Cartan matrix of G). Explicit and direct relationship may become more desirable if one wants to find an expression of the invariant in its natural generality (i.e. uniformly for all G/H) rather than for special cases. We present both a formula and an algorithm, which evaluate the number $a^w_{u,v}$ in term of Cartan numbers of G.

Assume that $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$, $\beta_i \in \Delta$, is a reduced decomposition of an $w \in \overline{W}$, and let $A_w = (a_{i,j})_{k\times k}$ be the associated Cartan matrix. For a subset $L = [i_1, \cdots, i_r] \subseteq [1, \cdots, k]$ we put $\mid L \mid = r$ and set

$$\sigma_L = \sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_r}} \in W; \quad x_L = x_{i_1} \cdots x_{i_r} \in \mathbb{Z}[x_1, \cdots, x_k].$$

Our solution to the problem is

Theorem. If $u, v \in \overline{W}$ with $l(w) = l(u) + l(v)$, then

$$a^w_{u,v} = T_{A_w}[(\sum_{\substack{|L|=l(u)\\ \sigma_L = u}} x_L)(\sum_{\substack{|K|=l(v)\\ \sigma_K = v}} x_K)],$$

where $L, K \subseteq [1, \cdots, k]$.

The proof of the Theorem will be developed in such a way as to suggest its analogue for multiplying generalized Schubert classes in the focal manifolds of isoparametric submanifolds. In particular, the Theorem is valid for the focal manifolds of an isoparametric submanifold with equal multiplicities 2, in which the classical flag manifolds G/H are special cases (cf. **7.2**, **7.5** and [HPT]).

In [Bi] S. Billey obtained a recurrence [Bi, (5.5)] that can be used to derive an expression for $a^w_{u,v}$ by using all of the quantities $\xi^v(t) \mid_\alpha$, $\xi^s(t) \mid_\alpha$ and $\pi_t \mid_\alpha$ with $u \leq s < t \leq w$, where the $\xi^s(t)$ and the π_t for $t, s \in W$ are certain polynomials in the simple roots of G defined respectively by Kostant and Kumar in [KK] and by Billey in [Bi], and where $\mid_\alpha$ means evaluating the polynomials at the regular point α. In view of [Bi, Theorem 4], our theorem expresses $a^w_{u,v}$ only in terms of the data required to describe $\xi^u(w)$ and $\xi^v(w)$.

Example 3. Let W be the Weyl group of G_2. Continuing from Example 1 we express the product $P_u P_v$ in $H^*(G_2/T)$ in terms of Schubert classes, where u, $v \in W$ are specified by the vectors $u(\alpha)$, $v(\alpha) \in L(T)$ in the figure. We note that $l(u) = 3$, $l(v) = 2$ and w, $w' \in W$ are the only elements with length $l(u) + l(v) = 5$.

Referring to the reduced decomposition $w = \sigma_{\beta_2} \circ \sigma_{\beta_1} \circ \sigma_{\beta_2} \circ \sigma_{\beta_1} \circ \sigma_{\beta_2}$ obtained in Example 1, the solutions in $L \subset [1, 2, 3, 4, 5]$ to the equations $\sigma_L = u$, $\mid L \mid = 3$ are

$$L = (1, 2, 3), (1, 2, 5), (1, 4, 5), (3, 4, 5);$$

and the solutions in $K \subset [1, 2, 3, 4, 5]$ to the equations $\sigma_K = v$, $\mid K \mid = 2$ are

$$K = (2, 3), (2, 5), (4, 5).$$

Using the Theorem we compute

$$\begin{aligned} a^w_{u,v} &= T_{A_w}[(x_1x_2x_3 + x_1x_2x_5 + x_1x_4x_5 + x_3x_4x_5)(x_2x_3 + x_2x_5 + x_4x_5)] \\ &= 2 + 2T_{A_w}(x_1x_2x_4x_5^2) + T_{A_w}(x_1x_4^2x_5^2) \\ &\quad + T_{A_w}(x_2x_3^2x_4x_5) + T_{A_w}(x_2x_3x_4x_5^2) + T_{A_w}(x_3x_4^2x_5^2), \end{aligned}$$

where the second equality follows from the additivity of T_{A_w} and an application of Corollary 1. With the matrix A_w being determined in Example 1, we find that

$$\begin{array}{ccccc} T_{A_w}(x_1x_2x_4x_5^2) & T_{A_w}(x_1x_4^2x_5^2) & T_{A_w}(x_2x_3^2x_4x_5) & T_{A_w}(x_2x_3x_4x_5^2) & T_{A_w}(x_3x_4^2x_5^2) \\ 1 & -2 & 1 & -1 & -1 \end{array}.$$

Consequently, $a^w_{u,v} = 1$.

Similarly, from the reduced decomposition $w\prime = \sigma_{\beta_1} \circ \sigma_{\beta_2} \circ \sigma_{\beta_1} \circ \sigma_{\beta_2} \circ \sigma_{\beta_1}$ we find

$$\sum_{\substack{|L|=l(u) \\ \sigma_L(\alpha)=u(\alpha)}} x_L = x_2x_3x_4, \qquad \sum_{\substack{|K|=l(v) \\ \sigma_K(\alpha)=v(\alpha)}} x_K = x_1x_2 + x_1x_4 + x_3x_4.$$

From the Theorem we get

$$a^{w'}_{u,v} = T_{A_w}[(x_2x_3x_4)(x_1x_2 + x_1x_4 + x_3x_4)] = 0 \text{ (by Corollary 1)}.$$

Summarizing, $P_uP_v = P_w$.

Remark 1. It follows from intersection theory that the coefficients $a^w_{u,v}$ are always non-negative. It is therefore attempted to have a method to compute the number without cancellation involved (i.e. *positively multiplying Schubert classes*). From the computation in Example 3 one finds that our method is not positive. However, the geometric reason behind this phenomenon can be easily clarified. This will be discussed in subsection 7.6.

2.4. The algorithm. In concrete situations one prefers to see the practical value of $a^w_{u,v}$ rather than the closed formula, for this could reveal in a direct way the intersection multiplicities of X_u with X_v in the variety X_w. For this purpose the Theorem does indicate an effective algorithm to evaluate $a^w_{u,v}$, as the following recipe shows (see also Examples 1-3).

(1) starting from the Cartan matrix of G, a program to enumerate all elements in a coset $\overline{W}$ of the Weyl group W by their *minimal reduced decompositions* is available in [DZZ];

(2) for an $w \in \overline{W}$ with a reduced decomposition, the corresponding Cartan matrix A_w can be read directly from Cartan matrix of G (cf. Example 1);

(3) for an $w \in \overline{W}$ with a reduced decomposition $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$ and an $u \in \overline{W}$ with $l(u) = r < k$, the solutions in the subsequence $[i_1, \cdots, i_r] \subseteq [1, \cdots, k]$ to the equation $\sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_r}} = u$ in W agree with the solutions to the vector equation $\sigma_{\beta_{i_1}} \circ \cdots \circ \sigma_{\beta_{i_r}}(\alpha) = u(\alpha)$ in the linear space $L(T)$, where $\alpha \in L(T)$ is a fixed regular point;

(4) the evaluation the operator T_{A_w} on a polynomial can be easily programmed (cf. Example 2).

Combining the ideas explained above, a program to compute the numbers $a^w_{u,v}$ has been compiled [DZ$_2$]. It uses Cartan matrix as the only input and

computations in various flag manifolds G/H can be performed by the single program.

As for the efficiency of the program, we refer the reader to the computational results tabulated in [DZ_1]. They were produced by a similar program that implements Steenrod operations on Schubert classes.

2.5. Arrangement of the paper. The rest sections of the paper are so arranged. Section 3 develops preliminary results from algebraic topology. We recall from [Du_2] the cohomology of twisted product of 2-spheres (Lemma 3.3), and the enumerative formula on these manifolds (Lemma 3.4). In particular, we introduce *divided differences* for spherical represented involutions, and their basic properties are established in Lemma 3.2.

We shall see in Section 4 that, by resorting to the geometry of the adjoint representation, Bott-Samelson cycles in the space G/T appears as certain twisted product of 2-spheres that are parameterized by ordered sequences of roots, and the divided differences on the integral cohomology of G/T arise naturally from the geometric fact that the involution on G/T corresponding to a root is spherical representable. After determining the induced action of Bott-Samelson cycles (corresponding to a sequence of simply roots) on Schubert classes in Lemma 5.1 (Section 5), the Theorem is established in Section 6.

In many literatures ranging from topology, algebraic and differential geometry to representation theory, one finds analogues of the terminologies that we work with, such as Bott-Samelson cycles (or schemes), divided differences and Schubert varieties, but with seemingly different appearances. In particular, Schubert varieties were originally introduced and extensively studied in the context of algebraic geometry, but we will work with Lie groups in the real compact form so that our method are ready to apply to general situations (cf. **7.5**). In order to merge our presentations into the existing literatures and, at the same time, not to interrupt our exposition, Section 7, entitled *Historical remarks*, is devoted to recall those historical events illustrating the readiness and necessity of the conceptual development in our paper, and will be referred to from time to time.

Finally, a brief account for the method of the proof. In 1973 Hansen discovered that the celebrated K-cycles on the flag manifold G/T constructed by Bott and Samelson in 1958 (cf. [BS_2] or **7.1**) provided a degree 1 map g_w from a twisted products Γ_w of 2-spheres onto the Schubert variety X_w (cf. [Han] or **7.4**). This suggests that the intersection product in X_w can be translated as part of the intersection product in Γ_w via the homomorphism induced by g_w. However, the latter is much easier to work with for the following reasons (cf. Lemma 3.3 and Lemma 3.4 in Section 3).

a) the space Γ_w admits a natural cell decomposition with each cell, again, *a twisted product of* 2-*spheres*;

b) the cohomology of Γ_w is a polynomial ring $\mathbb{Z}[x_1, \cdots, x_k]$ generated by x_i's in dimension 2, subject to relations occurring *only in dimension* 4;

c) the intersection product in Γ_w is handled by a triangular operator T_A. Therefore, the intersection multiplicity $a^w_{u,v}$ in question can be calculated by computations in a space like Γ_w, rather than in the Schubert variety X_w itself.

3 Preliminaries in topology

In this paper all homologies (resp. cohomologies) will have integer coefficients unless otherwise stated. If $f : X \to Y$ is a continuous map between two topological spaces, f_* (resp. f^*) is the homology (resp. cohomology) map induced by f. Write S^r for the r-dimensional sphere. If M is an oriented closed manifold (resp. a connected projective variety) $[M] \in H_{\dim M}(M)$ stands for the orientation class. The Kronecker pairing, between cohomology and homology of a space X, will be denoted by $<,>: H^*(X) \times H_*(X) \to \mathbb{Z}$.

3.1. Sphere bundle with a cross section. Let $p : E \to M$ be a smooth, oriented r-sphere bundle over an oriented manifold M which has a cross section $s : M \to E$. Let the normal bundle ξ of the embedding s be oriented by p, and let $e \in H^r(M)$ be the Euler class of ξ with respect to this orientation.

The integral cohomology $H^*(E)$ can be described as follows. Denote by $i : S^r \to E$ for the fiber inclusion of p over a point $z \in M$, and write by $J : E \to E$ for the involution given by the antipodal map in each fiber sphere. We have the following result from [Du$_2$, Lemma 4].

Lemma 3.1. *There exists a unique class $x \in H^r(E)$ such that*

$$s^*(x) = 0 \in H^*(M) \text{ and } < i^*(x), [S^r] >= 1.$$

Furthermore

(1) $H^*(E)$, as a module over $H^*(M)$, has the basis $\{1, x\}$ subject to the relation $x^2 + p^*(e)x = 0$;

(2) the induced cohomology map J^* acts identically on the subset $\operatorname{Im} p^* \subset H^*(E)$ and

$$J^*(x) = (-1)^{r-1}x - p^*(e).$$

Remark 2. If r is odd, $2e = 0$.

3.2. Divided difference of a spherical represented involution. A self-map σ of a manifold M is called *an involution* if $\sigma^2 = id : M \to M$. An *r-spherical representation* of the involution $(M; \sigma)$ is a system $f : (E; J) \to (M; \sigma)$ in which

(1) E is the total space of an oriented r-sphere bundle $p : E \to M$ with a cross section s;

(2) f is a continuous map $E \to M$ that satisfies the following two constrains

(3.1) $\quad f \circ s = id : M \to M$; and

(3.2) $\quad f \circ J = \sigma \circ f : E \to M$,

where $J : E \to E$ is the involution on E given by the antipodal map in the fibers.

In view of the $H^*(M)$-module structure of $H^*(E)$ specified by Lemma 3.1, an r-spherical representation f of the involution (M, σ) gives rise to an additive operator $\theta_f : H^m(M) \to H^{m-r}(M)$ of degree $-r$ that is characterized uniquely as follows.

The induced homomorphism $f^* : H^*(M) \to H^*(E)$ satisfies

(3.3) $$f^*(z) = p^*(z) + p^*(\theta_f(z))x$$

for all $z \in H^*(M)$.

Useful properties of θ_f can be derived directly from its definition (3.3).

Lemma 3.2. *Let $e \in H^r(M)$ be as that in Lemma 3.1. We have*

(1) $\sigma^* = Id - e\theta_f : H^*(M) \to H^*(M)$;

(2) $\theta_f(z_1 z_2) = \theta_f(z_1)z_2 + \sigma^*(z_1)\theta_f(z_2)$, $z_1, z_2 \in H^*(M)$.

(3) if r is even, then $\theta_f(e) = 2(\in H^0(M) = \mathbb{Z})$; if r is odd, then $\theta_f(e) = 0$;

(4) if r is even, $2\theta_f \circ \theta_f = 0 : H^*(M) \to H^*(M)$.

Proof. Since f^* is a ring map, we have

$$\begin{aligned} f^*(z_1 z_2) &= [p^*(z_1) + p^*(\theta_f(z_1))x][p^*(z_2) + p^*(\theta_f(z_2))x] \\ &= p^*(z_1 z_2) + p^*[\theta_f(z_1)z_2 + z_1\theta_f(z_2)]x + p^*[\theta_f(z_1)\theta_f(z_2)]x^2 \\ &= p^*(z_1 z_2) + p^*[\theta_f(z_1)z_2 + z_1\theta_f(z_2) - e\theta_f(z_1)\theta_f(z_2)]x, \end{aligned}$$

where the last equality is obtained from $x^2 = -p^*(e)x$ (because of the relation $x^2 + p^*(e)x = 0$). Comparing this with (3.3) yields

(3.4) $$\begin{aligned} \theta_f(z_1 z_2) &= \theta_f(z_1)z_2 + z_1\theta_f(z_2) - e\theta_f(z_1)\theta_f(z_2) \\ &= \theta_f(z_1)z_2 + (z_1 - e\theta_f(z_1))\theta_f(z_2). \end{aligned}$$

Applying J^* to (3.3) gives

$$J^* f^*(z) = p^*(z) + p^*(\theta_f(z))J^*(x)$$

by (2) of Lemma 3.1. From $J^* f^* = f^*\sigma^*$ (by (3.2)) and $J^*(x) = (-1)^{r-1}x - p^*(e)$ (by (2) of Lemma 3.1) we get

$$f^*(\sigma^*(z)) = p^*(z - e\theta_f(z)) + p^*((-1)^{r-1}\theta_f(z))x.$$

Comparing this with (3.3) gives rise to

(3.5) $\quad \sigma^*(z) = z - e\theta_f(z)$; and

(3.6) $\quad \theta_f(\sigma^*(z)) = (-1)^{r-1}\theta_f(z)$.

(1) is verified by (3.5). Combining (3.4) with (3.5) shows (2).

Substituting (3.5) in the left hand side of (3.6) and rewriting the resulting equation by using (3.4) gives

(3.7) $$(1 + (-1)^r)\theta_f(z) = \theta_f(e)\theta_f(z) + (e - \theta_f(e)e)\theta_f(\theta_f(z)).$$

Taking $z = e$ in (3.7) and noting that $\theta_f(\theta_f(e)) \in H^{-r}(M) = 0$ we get the equation $(1 + (-1)^r)\theta_f(e) = \theta_f(e)^2$ in $H^0(M) = \mathbb{Z}$. This proves (3).

Now (3.7) becomes $e\theta_f(\theta_f(z)) = 0$ for all $z \in H^*(M)$ by (3). Consequently, taking $z = \theta_f(u)$ in (3.5) yields $\sigma^*(\theta_f(u)) = \theta_f(u)$. Now (3.6) implies that if r is

even, $2\theta_f(\theta_f(u)) = 0$ for all $u \in H^*(M)$. This verifies (4), hence completes the proof of Lemma 3.2.

We observe from (1) of Lemma 3.2 that, for any $z \in H^*(M)$, the difference $z - \sigma^*(z) \in H^*(M)$ is always divisible by e with quotient $\theta_f(z)$.

Definition 3. The operator θ_f is called the *divided difference* of the spherical representation f of the involution (M, σ) (Compare with the discussion in **7.5**).

3.3. Twisted product of 2-spheres. The following definition singles out a class spaces in which we will be particularly interested.

Definition 4. A smooth manifold M is called an *oriented twisted product of 2-spheres of rank k*, denoted by $M = \underset{1\leq i\leq k}{\propto} S^2$, if there is a tower of smooth maps

$$M = M_k \overset{p_{k-1}}{\to} M_{k-1} \overset{p_{k-2}}{\to} \cdots \overset{p_2}{\to} M_2 \overset{p_1}{\to} M_1$$

in which

1) M_1 is diffeomorphic to S^2 with an orientation;
2) p_i is the projection of an oriented smooth S^2 bundle over M_i;
3) p_i has a fixed cross section s_i, $1 \leq i \leq k-1$.

Let $M = \underset{1\leq i\leq k}{\propto} S^2$ be a twisted product of 2-spheres of rank k. The cross sections s_i yield a sequence of embeddings

$$M_1 \overset{s_1}{\to} M_2 \overset{s_2}{\to} \cdots \overset{s_{k-1}}{\to} M_k = M.$$

In view of this we shall make no notational distinction between a subspace $N \subset M_i$ and its image under s_i in M_j, $j \geq i$. Take a base point $x_0 \in M_1$ and let each M_i has x_0 as its base point.

For a subset $L = [i_1, \cdots, i_r] \subseteq [1, \cdots, k]$ a smooth submanifold

$$S(L) \subset M_{i_k} \subset M_{i_k+1} \subset \cdots \subset M$$

can be introduced inductively as follows.

1) $S(1) = M_1$;
2) if $i > 1$, $S(i) \subset M_i$ is the fiber sphere of p_{i-1} over the base point;
3) assume that $S(L') \subset M_{i_{r-1}}$, where $L' = [i_1, \cdots, i_{r-1}]$, has been defined and consider the case $L = [i_1, \cdots, i_{r-1}, j]$. Then $S(L) \subset M_j$ is the total space of the restricted bundle of $p_{j-1} : M_j \to M_{j-1}$ to the subspace $S(L') \subset M_{j-1}$. The natural bundle map over the inclusion $S(L') \subset M_{i_{r-1}} \to M_{j-1}$ gives rise to the desired embedding $S(L) \subseteq M_j \subseteq M$.

The integral homology (resp. cohomology) of an $M = \underset{1\leq i\leq k}{\propto} S^2$ can be described as follows. Consider the normal bundle $\pi_i : E_i \to M_{i-1}$ of the embedding $s_{i-1} : M_{i-1} \to M_i$, $2 \leq i \leq k$. It is a 2-plane bundle with a natural orientation inherited from that on p_{i-1}. Let $e_i' \in H^2(M_{i-1})$ be the Euler class of π_i. We set $e_1 = 0$ and for $i \geq 2$

$$e_i = (p_{i-1} \circ p_i \circ \cdots \circ p_{k-1})^* e_i' \in H^2(M).$$

Lemma 3.3. *For an $M = \underset{1\leq i\leq k}{\propto} S^2$ let $[S(L)] \in H_{2|L|}(M)$ be the fundamental class of the cycle $S(L) \subset M$. Then*

(1) the set $\{[S(L)] \mid L \subseteq [1, \cdots, k]\}$ of homology classes is an additive basis for the graded $\mathbb{Z}$-module $H_*(M)$.

Further, let $\{x_L \in H^{2|L|}(M) \mid L \subseteq [1, \cdots, k]\}$ be the basis of $H^*(M)$ Kronecker dual to the basis $\{[S(L)] \mid L \subseteq [1, \cdots, k]\}$ in homology. Then

(2) $x_L = x_{i_1} \cdots x_{i_r}$ if $L = [i_1, \cdots, i_r]$;

(3) $H^*(M) = \mathbb{Z}[x_1, \cdots, x_k]/ < x_i^2 + e_i x_i; 1 \leq i \leq k >$, where e_i is a polynomial in $x_1, \cdots, x_{i-1}$.

Proof. By setting $T_r(M) = \underset{|L|=r}{\cup} S(L) \subseteq M$, $1 \leq r \leq k$, we get a filtration of subspaces

(3.8) $$T_1(M) \subset T_2(M) \subset \cdots \subset T_k(M) = M.$$

Moreover, the subspace $T_r(M) \backslash T_{r-1}(M)$ consisting of $2r$-dimensional open cells in one-to-one correspondence with the subset $L \subseteq [1, \cdots, k]$ with $\mid L \mid = r$. This implies that (3.8) dominates M by a cell complex with only even dimensional cells. This proves (1).

Assertions (2) and (3) follow easily from (1) of Lemma 3.1 together with induction on k.

For degree reasons the polynomials e_i are all homogeneous of degree 1 in $x_1, \cdots, x_{i-1}$, i.e.

$$e_i = a_{1,i} x_1 + \cdots + a_{i-1,i} x_{i-1},\ a_{i,j} \in \mathbb{Z},\ 1 \leq i \leq k.$$

Definition 5. With $a_{i,j} = 0$ for $i \geq j$ being understood, the strictly upper triangular matrix $A = (-a_{i,j})_{k\times k}$ is called *the structure matrix* of $M = \underset{1\leq i\leq k}{\propto} S^2$.

Remark 3. It was shown in [Du_2, Proposition 1] that any strictly upper triangular matrix A of rank k can be realized as the structure matrix of an $M = \underset{1\leq i\leq k}{\propto} S^2$.

3.4. Integration along a twisted product of 2-spheres. Recall from Lemma 3.3 that the integral cohomology of an $M = \underset{1\leq i\leq k}{\propto} S^2$ can be specified as a quotient of a free polynomial ring

$$H^*(M) = \mathbb{Z}[x_1, \cdots, x_k]/ < x_i^2 + e_i x_i; 1 \leq i \leq k >.$$

Write $\mathbb{Z}[x_1, \cdots, x_k]^{(r)}$ for the subset of all homogeneous polynomials of degree r in $\mathbb{Z}[x_1, \cdots, x_k]$ and let $p_M : \mathbb{Z}[x_1, \cdots, x_k]^{(r)} \to H^{2r}(M)$ be the obvious quotient map. Consider the additive correspondence $\int_M : \mathbb{Z}[x_1, \cdots, x_k]^{(k)} \to \mathbb{Z}$ defined by

$$\int_M h =< p_M(h), [M] >,$$

where $[M] \in H_{2k}(M) = \mathbb{Z}$ is the orientation class.

As being indicated by the notation, the operator $\int_M$ can be interpreted as "*integration along* M" in De Rham theory. In view of discussions in [Du_1], the problem of effective computation of the intersection product in M asks an effective algorithm to evaluate $\int_M$. The idea of structure matrix of M, together

with the operator T_A introduced in **2.2**, is useful in presenting such an algorithm. The following result was shown in [Du$_2$, Proposition 2].

Lemma 3.4. If M has the structure matrix $A = (a_{ij})_{k\times k}$, then

$$\int_M = T_A : \mathbb{Z}[x_1, \cdots, x_k]^{(k)} \to \mathbb{Z}.$$

4 Geometry from adjoint representation

Since the point $\alpha \in L(T)$ is regular, the adjoint representation $Ad : G \to L(G)$ yields a smooth embedding

$$\varphi : G/T \to L(G) \qquad \text{by } \varphi(gT) = Ad_g(\alpha).$$

In this way G/T becomes a submanifold of the Euclidean space $L(G)$.

4.1. Preliminaries. Let Φ^+ be the set of positive roots relative to α (cf. **2.1**). Assume that the Cartan decomposition of the Lie algebra $L(G)$ relative to $T \subset G$ is

$$L(G) = L(T) \oplus_{\beta\in\Phi^+} F_\beta,$$

where F_β is the root space, viewed as a real 2-plane, belonging to the root $\beta \in \Phi^+$ ([Hu, p.35]). Let $[,]$ be the Lie bracket on $L(G)$. We make a simultaneous choice of an orientation on each F_β by the following rule: take a $v \in F_\beta\backslash\{0\}$ and let $v\prime \in F_\beta$ be such that $[v, v'] = \beta$. Then F_β is oriented by the ordered basis $\{v, v'\}$. Clearly this orientation on F_β is independent of a specific choice of v.

We quote from [HPT, p.426-427] for relevant information concerning the geometries of the submanifold $G/T \subset L(G)$.

(4.1) The subspaces $\oplus_{\beta\in\Phi^+} F_\beta$ and $L(T)$ of $L(G)$ are tangent and normal to G/T at α respectively;

(4.2) The tangent bundle to G/T has a canonical orthogonal decomposition into the sum of m integrable 2-plane bundles $\oplus_{\beta\in\Phi^+} E_\beta$ with $E_\beta(\alpha) = F_\beta$.

In general, $E_\beta(y) = Ad_g(F_\beta)$ if $y = Ad_g(\alpha) \in G/T$.

(4.3) The leaf of the integrable subbundle E_β through a point $y \in G/T$, denoted by $S(y; \beta)$, is a 2-sphere.

In view of (4.2) and (4.3) we adopt the following convention.

Convention. Each bundle E_β, $\beta \in \Phi^+$, is oriented such that the identification $E_\beta(\alpha) = F_\beta$ is orientation preserving. Let the 2-sphere $S(y; \beta)$ have the orientation inherited from that on its tangent plane $E_\beta(y)$ at $y \in S(y; \beta)$.

By considering G/T as a submanifold in the Euclidean space $L(G)$ via the embedding φ, the geometric features singled out in (4.2) and (4.3) will dominate all of our geometric constructions and computations in this section.

4.2. Realization of roots as homology and cohomology classes. We start by relating the set Φ of roots with certain 2-dimensional homology (resp. cohomology) classes of G/T.

For a $\beta \in \Phi^+$ write by $e_\beta \in H^2(G/T)$ for the Euler class of the oriented bundle E_β, and let $\varphi_\beta : S(y;\beta) \to G/T$ be inclusion of the leaf sphere (cf. (4.3)).

Lemma 4.1. *The Kronecker paring $H^2(G/T) \times H_2(G/T) \to \mathbb{Z}$ can be expressed in term of Cartan numbers as*

$$< e_\gamma, \varphi_{\beta *}[S(y;\beta)] >= \beta \circ \gamma,\ \beta, \gamma \in \Phi,$$

where $[S(y;\beta)] \in H^2(S(y;\beta))(= \mathbb{Z})$ is the orientation class.

Remark 4. The class $\varphi_{\beta *}[S(y;\beta)] \in H_2(G/T)$ is independent of the choice of the point $y \in G/T$. Let $\varepsilon : [0,1] \to G/T$ be a path that joins y to some y'. Then the continuous one-parameter family of 2-spheres $S(\varepsilon(t);\beta) \subset G/T$, $t \in [0,1]$ is an isotopy from $S(y;\beta)$ to $S(y';\beta)$.

The Weyl group W of G, acting as isometries of the Cartan subalgebra $L(T)$, has the effect to permute the set Φ of roots [Hu]. On the other hand, via the embedding φ, the canonical action of W on G/T [BS$_2$] is given by

(4.4) $$w(x) = Ad_g(w(\alpha)),\ x = Ad_g(\alpha) \in G/T,\ w \in W.$$

Let $w^* : H^*(G/T) \to H^*(G/T)$ be the induced action on cohomology.

Lemma 4.2. *For an $w \in W$ one has*

(1) $w(S(y;\beta)) = S(w(y); w(\beta))$;

(2) $w^*(E_\beta) = E_{w^{-1}(\beta)}$; and

(3) $w^*(e_\beta) = e_{w^{-1}(\beta)}$.

Proof. In term of the W-action on the set Φ of roots, the induced bundle $w^*(E_\beta)$ is $E_{w^{-1}(\beta)}$ (cf. [HTP, **1.6**]). This shows (2). Assertion (3) comes now from the naturality of Euler classes.

Item (1), which follows also from (2) (and (4.3)), indicates that the action of w on G/T has the effect to carry the leaf sphere through y corresponding to a root β diffeomorphically onto the leaf sphere through $w(y)$ corresponding to the root $w(\beta)$.

4.3. **2-spherical represented involutions.** Each root $\beta \in \Phi^+$ gives rise to an involution $\sigma_\beta : G/T \to G/T$ in the fashion of (4.4), and defines also the subspace

$$S(\beta) = \{(y, y_1) \in G/T \times G/T \mid y_1 \in S(y;\beta)\}.$$

Projection $p_\beta : S(\beta) \to G/T$ onto the first factor is easily seen to be a 2-sphere bundle projection (with the leaf sphere $S(y;\beta)$ as the fiber over $y \in G/T$). The map $s_\beta : G/T \to S(\beta)$ by $s_\beta(y) = (y, y)$ furnishes p_β with a ready-made cross section.

Let J_β be the involution on $S(\beta)$ given by the antipodal map on each fiber sphere and let $f_\beta : S(\beta) \to G/T$ be the projection onto the second factor. Then, as is clear,

$$f_\beta \circ s_\beta = id : G/T \to G/T;$$
$$f_\beta \circ J_\beta = \sigma_\beta \circ f_\beta : S(\beta) \to G/T.$$

That is, *the map $f_\beta : (S(\beta), J_\beta) \to (G/T, \sigma_\beta)$ is a 2-spherical representation of the involution $(G/T, \sigma_\beta)$* (cf. **3.2**).

Lemma 4.3. *Assume the same setting as the above.*

1) As a $H^*(G/T)$-module, the cohomology of $S(\beta)$ is given by

$$H^*(S(\beta)) = H^*(G/T)[1,x]/ < x^2 + e_\beta x >,$$

in which $x \in H^2(S(\beta))$ is uniquely characterized by $s_\beta^*(x) = 0$ and

$$< i^*(x), [S(y;\beta)] >= 1 \text{ (for all } y \in G/T\text{)},$$

where $i : S(y;\beta) \to S(\beta)$ is the inclusion of the fiber over $y \in G/T$.

2) Write $\theta_\beta : H^*(G/T) \to H^*(G/T)$ to denote the divided difference associated to the 2-spherical representation f_β of the involution $(G/T, \sigma_\beta)$. Then

(1) for any $\gamma \in \Phi$, $\theta_\beta(e_\gamma) = \beta \circ \gamma (\in H^0(G/T) = \mathbb{Z})$;

(2) $\theta_\beta \circ \theta_\beta = 0$;

(3) $\sigma_\beta^* = Id - e_\beta\theta_\beta : H^*(G/T) \to H^*(G/T)$;

(4) $\theta_\beta(z_1z_2) = \theta_\beta(z_1)z_2 + \sigma_\beta^*(z_1)\theta_\beta(z_2)$, $z_1, z_2 \in H^*(G/T)$, and

(5) for any $w \in W$, $\theta_{w(\beta)} = (w^{-1})^*\theta_\beta w^*$.

Proof. Since the normal bundle of the embedding s_β is E_β, one has the relation $x^2 + e_\beta x = 0$ by (1) of Lemma 3.1. This verifies 1).

In 2) properties (3) and (4) corresponds to the items (3) and (4) in Lemma 3.2. (2) follows from (2) of Lemma 3.2 since $H^*(G/T)$ is torsion free [BS$_1$]. It remains to show (1) and (5).

Property (1) is verified by

$\beta \circ \gamma =< e_\gamma, \varphi_{\beta *}[S(y;\beta)] >$(by Lemma 4.1)

$=< \varphi_\beta^*(e_\gamma), [S(y;\beta)] >$ (by the naturality of $<,>$)

$=< i^* \circ f_\beta^*(e_\gamma), [S(y;\beta)] >$ (since $\varphi_\beta = f_\beta \circ i$)

$=< i^*(p_\beta^*(e_\gamma) + p_\beta^*(\theta_\beta(e_\gamma))x), [S(y;\beta)] >$ (the definition of θ_β in (3.3))

$= \theta_\beta(e_\gamma)$ (by 1) of this Lemma),

where in the last equality we have applied the standard fact that

(4.5) *the composition $i^* \circ p_\beta^* : H^r(G/T) \to H^r(S(y;\beta))$ is zero in degree $r > 0$, and is an isomorphism $\mathbb{Z} \to \mathbb{Z}$ if $r = 0$.*

Finally we show (5). For an $w \in W$ the self diffeomorphism $w \times w : G/T \times G/T \to G/T \times G/T$ restricts to a diffeomorphism $\widetilde{w} : S(\beta) \to S(w(\beta))$ (by (1) of Lemma 4.2) that fits into the following two commutative diagrams

$$\begin{array}{ccc} S(\beta) & \xrightarrow{\widetilde{w}} & S(w(\beta)) \\ p_\beta \downarrow & & \downarrow p_{w(\beta)} \\ G/T & \xrightarrow{w} & G/T \end{array} \quad \text{and} \quad \begin{array}{ccc} S(\beta) & \xrightarrow{\widetilde{w}} & S(w(\beta)) \\ f_\beta \downarrow & & \downarrow f_{w(\beta)} \\ G/T & \xrightarrow{w} & G/T. \end{array}$$

From the definition of θ_β in (3.3) we have

$$f_\beta^*(w^*(z)) = p_\beta^*(w^*(z)) + p_\beta^*(\theta_\beta(w^*(z)))x,\ z \in H^*(G/T).$$

On the other hand

$f_\beta^*(w^*(z)) = \widetilde{w}^* f_{w(\beta)}^*(z)$ (by the commutivity of the second diagram)

$= \widetilde{w}^*(p_{w(\beta)}^*(z) + p_\beta^*[\theta_{w(\beta)}(z)]x')$ (by the definition of $\theta_{w(\beta)}$)

$= p_\beta^*(w^*(z)) + p_\beta^*(w^*[\theta_{w(\beta)}(z)])x$ (by the commutivity of the first diagram).

Comparing the coefficients of $x \in H^*(S(\beta))$ in the above two expressions of $f_\beta^*(w^*(z))$ yields $w^*\theta_{w(\beta)} = \theta_\beta w^*$. This shows (5).

Lemma 4.4. *Given an ordered sequence $\beta_1, \cdots, \beta_k \in \Delta$ of simple roots we put $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$, $\theta_{(\beta_1,\cdots,\beta_k)} = \theta_{\beta_1} \circ \cdots \circ \theta_{\beta_1}$.*

(1) if $l(w) < k$, then $\theta_{(\beta_1,\cdots,\beta_k)} = 0$;

(2) if $l(w) = k$, then $\theta_{(\beta_1,\cdots,\beta_k)}$ depends only on w and not on the ordered sequence $\beta_1, \cdots, \beta_k \in \Delta$. In this case we put $\theta_w = \theta_{(\beta_1,\cdots,\beta_k)}$.

Proof. Furnished with properties (1)-(5) in Lemma 4.3 (in analogue with 3.3 Lemma in [BGG]), an argument parallel to the proof of [BGG, 3.4 Theorem] verifies Lemma 4.4. To do so one needs only to replace $\mathfrak{b}_{\mathbb{Q}}^* = \mathfrak{b}_{\mathbb{Z}}^* \otimes \mathbb{Q}$ in [BGG] by $H^2(G/T)$, where $\mathfrak{b}_{\mathbb{Z}}^*$ is the group of weights of G [Hu. p.67], and to resort to certain properties of Weyl groups from [BGG; §2]. For brevity we omit the details (see also Proposition 6 in **7.5**).

4.4. Bott-Samelson cycles and their cohomologies. Iterating the construction of leaf spheres gives rise to Bott-Samelson cycles.

For a $y \in G/T$ and an ordered sequence $(\beta_1, \cdots, \beta_k)$ of k roots (in which repetitions like $\beta_i = \beta_j$ for some $1 \le i < j \le k$ may occur), we construct an oriented twisted product of 2-spheres $S(y; \beta_1, \cdots, \beta_k)$ together with a smooth map

$$\varphi_{\beta_1,\cdots,\beta_k} : S(y; \beta_1, \cdots, \beta_k) \to G/T$$

by induction on k. To start with, $S(y; \beta_1)$ is the leaf sphere specified by (4.3) with φ_{β_1} the natural inclusion $S(y; \beta_1) \subset G/T$. Assume next that the map $\varphi_{\beta_1,\cdots,\beta_{k-1}} : S(y; \beta_1, \cdots, \beta_{k-1}) \to G/T$ has been defined. Let

$$p_{k-1} : S(y; \beta_1, \cdots, \beta_k) \to S(y; \beta_1, \cdots, \beta_{k-1})$$

be the induced bundle of $p_{\beta_k} : S(\beta_k) \to G/T$ via $\varphi_{\beta_1,\cdots,\beta_{k-1}}$. The cross section s_{β_k} of p_{β_k} defines a cross section s_{k-1} of p_{k-1}. The obvious bundle map $\widehat{\varphi}_{\beta_1,\cdots,\beta_{k-1}}$ over $\varphi_{\beta_1,\cdots,\beta_{k-1}}$ followed by f_{β_k} gives rise to the desired map

$$\varphi_{\beta_1,\cdots,\beta_k} = f_{\beta_k} \circ \widehat{\varphi}_{\beta_1,\cdots,\beta_{k-1}}. \tag{4.6}$$

The final step in the above construction is illustrated by the diagram below.

$$\begin{array}{ccccc} S(y; \beta_1, \cdots, \beta_k) & \overset{\widehat{\varphi}_{\beta_1,\cdots,\beta_{k-1}}}{\longrightarrow} & S(\beta_k) & & \\ p_{k-1} \downarrow\uparrow s_{k-1} & & p_{\beta_k} \downarrow\uparrow s_{\beta_k} & \overset{f_{\beta_k}}{\searrow} & \\ S(y; \beta_1, \cdots, \beta_{k-1}) & \overset{\varphi_{\beta_1,\cdots,\beta_{k-1}}}{\longrightarrow} & G/T & \overset{f_{\beta_k} \circ s_{\beta_k} = id}{\longrightarrow} & G/T. \end{array} \tag{4.7}$$

Remark 5. Alternatively one has

$$S(y; \beta_1, \cdots, \beta_i) = \{(y_1, \cdots, y_i) \in G/T \times \cdots \times G/T \mid y_1 \in S(y; \beta_1), \cdots, y_i \in S(y; \beta_i)\}.$$

The projection p_{i-1} (resp. the cross section s_{i-1}) is given by

$$p_{i-1}(y_1, \cdots, y_i) = (y_1, \cdots, y_{i-1})$$

$$(\text{resp. } s_{i-1}(y_1, \cdots, y_{i-1}) = (y_1, \cdots, y_{i-1}, y_{i-1})).$$

The map $\varphi_{\beta_1,\cdots,\beta_i} : S(y; \beta_1, \cdots, \beta_i) \to G/T$ is seen to be

$$\varphi_{\beta_1,\cdots,\beta_i}(y_1, \cdots, y_i) = y_i.$$

Definition 6 (cf. **7.2**). The map $\varphi_{\beta_1,\cdots,\beta_k} : S(y;\beta_1,\cdots,\beta_k) \to G/T$ is called *the Bott-Samelson cycle* associated to the sequence $\beta_1,\cdots,\beta_k$ of roots.

For a subset $L = [i_1,\cdots,i_r] \subseteq [1,\cdots,k]$, consider the embedding $i_L : S(y;\beta_{i_1},\cdots,\beta_{i_r}) \to S(y;\beta_1,\cdots,\beta_k)$ by

$$i_L(z_{i_1},\cdots,z_{i_r}) = (y_1,\cdots,y_k),$$

where $y_t = z_{i_s}$ if $i_s \leq t < i_{s+1}$ (cf. Remark 5). Set

$$S(y;L) = \mathrm{Im}\{i_L : S(y;\beta_{i_1},\cdots,\beta_{i_r}) \to S(y;\beta_1,\cdots,\beta_k)\}.$$

Note that the tower of smooth maps

$$S(y;\beta_1,\cdots,\beta_k) \overset{p_{k-1}}{\to} S(y;\beta_1,\cdots,\beta_{k-1}) \overset{p_{k-2}}{\to} \cdots \overset{p_1}{\to} S(y;\beta_1),$$

together with the sections s_i, $1 \leq i \leq k-1$, furnishes the space $S(y;\beta_1,\cdots,\beta_k)$ with the structure of a twisted product of 2-spheres.

Lemma 4.5. *Let $[S(y;L)] \in H_{2|L|}(S(y;\beta_1,\cdots,\beta_k))$ be the fundamental class of the cycle $S(y;L) \subset S(y;\beta_1,\cdots,\beta_k)$. Then*

(1) the set $\{[S(y;L)] \mid L \subseteq [1,\cdots,k]\}$ constitutes an additive basis for the graded $\mathbb{Z}$-module $H_*(M)$.

Further, let $x_i \in H^2(S(y;\beta_1,\cdots,\beta_k))$,$1 \leq i \leq k$, be the classes Kronecker dual to the basis $\{[S(y;i)] \mid 1 \leq i \leq k\}$ of $H_2(S(y;\beta_1,\cdots,\beta_k))$. Then

(2) the structure matrix of $S(y;\beta_1,\cdots,\beta_k)$ (with respect to $x_1,\cdots,x_k$) is $A = (a_{ij})_{k\times k}$, where

$$a_{ij} = \begin{cases} -\beta_i \circ \beta_j \text{ if } i < j; \\ 0 \text{ if } i > j. \end{cases}$$

(3) the Kronecker pairing in $S(y;\beta_1,\cdots,\beta_k)$ is given by

$$< x_L, [S(y;K)] >= \delta_{L,K},\ L, K \subseteq [1,\cdots,k],$$

where $x_L = x_{i_1}\cdots x_{i_r}$ if $L = [i_1,\cdots,i_r]$.

Proof. (1) and (3) correspond, respectively, to the items (1) and (3) in Lemma 3.3. It remains to show (2).

For an $2 \leq j \leq k$ the normal bundle of the section $s_{j-1} : S(y;\beta_1,\cdots,\beta_{j-1}) \to S(y;\beta_1,\cdots,\beta_j)$ is seen to be the the induced bundle $\varphi^*_{\beta_1,\cdots,\beta_{j-1}} E_{\beta_j}$, whose Euler class $e_j \in H^2(S(y;\beta_1,\cdots,\beta_{j-1}))$ is $\varphi^*_{\beta_1,\cdots,\beta_{j-1}}(e_{\beta_j})$ by the naturality property of Euler classes. That is

$$e_j = \varphi^*_{\beta_1,\cdots,\beta_{j-1}}(e_{\beta_j}).$$

On the other hand, by assuming that the structure matrix of $S(y;\beta_1,\cdots,\beta_k)$ is $A = (a_{ij})_{k\times k}$, we have the expression

$$e_j = -a_{1,j}x_1 - \cdots - a_{j-1,j}x_{j-1}$$

by definition 5, Section 3. Since the x_t are Kronecker dual to the $[S(y;[t])$ we have, for $t < j$, that

$$\begin{aligned} -a_{t,j} &=< \varphi^*_{\beta_1,\cdots,\beta_{j-1}}(e_{\beta_j}), [S(y;[t])] > \\ &=< e_{\beta_j}, \varphi_{\beta_1,\cdots,\beta_{j-1}*} i_{[t]*}[S(y;\beta_t)] > \text{ (by the naturality of } <,>) \\ &=< e_{\beta_j}, \varphi_{\beta_t *}[S(y;\beta_t)] > \text{ (since } \varphi_{\beta_1,\cdots,\beta_{j-1}} \circ i_{[t]} = \varphi_{\beta_t}) \\ &= \beta_t \circ \beta_j \text{ (by Lemma 4.1).} \end{aligned}$$

This completes the proof of Lemma 4.5.

5 The induced action of a Bott-Samelson cycle

Given a sequence $\beta_1, \cdots, \beta_k$ of roots consider the induce ring map

$$\varphi^*_{\beta_1,\cdots,\beta_k} : H^*(G/T) \to H^*(S(y;\beta_1,\cdots,\beta_k))$$

of the Bott-Samelson cycle $\varphi_{\beta_1,\cdots,\beta_k}$. The product in $H^*(S(y;\beta_1,\cdots,\beta_k))$ is well understood (cf. Lemma 3.4 and Lemma 4.5). Our aim is to reduce calculations in the ring $H^*(G/T)$ (which has been posed to be in question) to that in $H^*(S(y;\beta_1,\cdots,\beta_k))$ via $\varphi^*_{\beta_1,\cdots,\beta_k}$.

Recall from Lemma 4.5 that $H^*(S(y;\beta_1,\cdots,\beta_k))$ has the additive basis $\{x_L \mid L \subseteq [1,\cdots,k]\}$. Therefore, for a $u \in H^{2r}(G/T)$, one has a unique expression

(5.1) $$\varphi^*_{\beta_1,\cdots,\beta_k}(u) = \sum_{|L|=r;L\subseteq [1,\cdots,k]} a_L(u)x_L,\ a_L(u) \in \mathbb{Z}.$$

The determination of $\varphi^*_{\beta_1,\cdots,\beta_k}$ amounts to find the $a_L(u)$.

In the special cases where $\beta_1, \cdots, \beta_k$ is a sequence of simple roots, the action of $\varphi^*_{\beta_1,\cdots,\beta_k}$ on Schubert classes P_w, $w \in W$, in G/T can be determined completely.

Lemma 5.1. If $\beta_1, \cdots, \beta_k$ is a sequence of simple roots, then the induced map $\varphi^*_{\beta_1,\cdots,\beta_k}$ satisfies

$$\varphi^*_{\beta_1,\cdots,\beta_k}(P_w) = (-1)^{l(w)} \sum_{|L|=l(w),\sigma_L=w} x_L.$$

This section is devoted to a proof of Lemma 5.1.

5.1. As the first step we express the coefficient $a_L(u)$ in (5.1) in terms of the divided differences θ_β. For a subset $L = [i_1, \cdots, i_r] \subseteq [1, \cdots, k]$ write θ_L for the composition $\theta_{\beta_{i_1}} \circ \cdots \circ \theta_{\beta_{i_r}}$.

Lemma 5.2. *If $u \in H^{2r}(G/T)$, then for $L \subseteq [1,\cdots,k]$ with $\mid L \mid= r$*

$$a_L(u) = \theta_L(u)\ (\in H^0(G/T) = \mathbb{Z}).$$

Proof. In term of the Kronecker pairing one has

$$\begin{aligned}
a_L(u) &=< \varphi^*_{\beta_1,\cdots,\beta_k}(u), [S(y;L)] > \text{ (by (3) of Lemma 4.5)}\\
&=< i^*_L \circ \varphi^*_{\beta_1,\cdots,\beta_k}(u), [S(y;\beta_{i_1},\cdots,\beta_{i_r})] > \text{ (by the naturality of } <,>)\\
&=< \varphi^*_{\beta_{i_1},\cdots,\beta_{i_r}}(u), [S(y;\beta_{i_1},\cdots,\beta_{i_r})] > \text{ (since } \varphi_{\beta_{i_1},\cdots,\beta_{i_r}} = \varphi_{\beta_1,\cdots,\beta_k} \circ i_L).
\end{aligned}$$

This reduces the proof to the special case $L = [1, \cdots, k]$. This will be done by induction on k. The case $k = 1$ is easily verified by

$$\begin{aligned}
\varphi^*_{\beta_1}(u) &= i^* \circ f^*_{\beta_1}(u) = i^*(p^*_{\beta_1}(u) + p^*_{\beta_1}(\theta_{\beta_1}(u))x)\\
&= \theta_{\beta_1}(u)x \text{ (cf. (4.5))}.
\end{aligned}$$

Assume, finally, that $L = [1, \cdots, k]$ (i.e. $u \in H^{2k}(G/T)$). We compute

$$\begin{aligned}
&\varphi^*_{\beta_1,\cdots,\beta_k}(u) = \widehat{\varphi}^*_{\beta_1,\cdots,\beta_{k-1}}(f^*_{\beta_k}(u)) \text{ (by (4.6))}\\
&= \widehat{\varphi}^*_{\beta_1,\cdots,\beta_{k-1}}(p^*_{\beta_k}(u) + p^*_{\beta_k}(\theta_{\beta_k}(u))x) \text{ (by the definition of } \theta_{\beta_k} \text{ in (3.3))}\\
&= p^*_{k-1}(\varphi^*_{\beta_1,\cdots,\beta_{k-1}}(u)) + p^*_{k-1}(\varphi^*_{\beta_1,\cdots,\beta_{k-1}}(\theta_{\beta_k}(u)))x_k \text{ (by the diagram (4.7))}\\
&= p^*_{k-1}(\varphi^*_{\beta_1,\cdots,\beta_{k-1}}(\theta_{\beta_k}(u)))x_k \ (\ \varphi^*_{\beta_1,\cdots,\beta_{k-1}}(u) \in H^{2k}(S(y;\beta_1,\cdots,\beta_{k-1})) = 0)\\
&= p^*_{k-1}(\theta_{[1,\cdots,k-1]}(\theta_{\beta_k}(u))x_1 \cdots x_{k-1}))x_k \text{ (by the inductive hypothesis)}\\
&= \theta_L(u)x_L.
\end{aligned}$$

This finishes the proof.

5.2. Bott-Samelson resolution of X_w. We refer to **7.3** and **Definition 7** in **7.4** for two equivalent geometric descriptions of Schubert varieties X_w, $w \in W$, in G/T. What is really relevant to us is the desingularization of X_w, rather than X_w itself, originated from Bott-Samelson in the following way (compare [BS$_2$, p.1000] with discussions in **7.1-7.4**).

Picture W as the W-orbit $\{w(\alpha) \in L(T) \mid w \in W\}$ of the regular point α. For an $w \in W$ let C_w be a straight line segment in $L(T)$ from the Weyl chamber containing α to $w(\alpha)$ that crosses the planes in $D(G)$ one at a time. Assume that they are met in the order $L_{\alpha_1}, \cdots, L_{\alpha_k}$, $\alpha_i \in \Phi^+$. We have $l(w) = k$ and

$$w = \sigma_{\alpha_k} \circ \cdots \circ \sigma_{\alpha_1},$$

where σ_α is the reflection of $L(T)$ in $L_\alpha \in D(G)$ (cf. [Han]).

In term of the sequence of positive roots $\alpha_1, \cdots, \alpha_k$ specified by the segment C_w we form the composed map

(5.2) $$\varphi_w = w \circ \varphi_{\alpha_1,\cdots,\alpha_k} : S(\alpha; \alpha_1, \cdots, \alpha_k) \to G/T,$$

where $\varphi_{\alpha_1,\cdots,\alpha_k}$ is the Bott-Samelson cycle associated to $\alpha_1, \cdots, \alpha_k$. The map φ_w may be appropriately termed as a *Bott-Samelson resolution of* X_w by the next result shown in [Du$_2$, Proposition 3].

Lemma 5.3. *The map φ_w is a degree 1 map onto X_w.*

¿From Lemma 5.3 we have, for a Schubert class $P_{w'} \in H^{2k}(G/T)$, that

$$\varphi_w^*(P_{w'}) = \delta_{w,w'} x_1 \cdots x_k.$$

On the other hand one has

$$\varphi_w^*(u) = \theta_{\alpha_1} \circ \cdots \circ \theta_{\alpha_k}[w^*(u)] x_1 \cdots x_k,\ u \in H^{2k}(G/T)$$

by Lemma 5.2. These imply that

Lemma 5.4. $\theta_{\alpha_1} \circ \cdots \circ \theta_{\alpha_k}[w^*(P_{w'})] = \delta_{w,w'}$.

5.3. The operator $\theta_w : H^{2l(w)}(G/T) \to H^0(G/T) = \mathbb{Z}$. Let $w \in W$ be with $l(w) = k$. Recall from Lemma 4.4 that if $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$ is reduced decomposition, then the composition

$$\theta_w = \theta_{\beta_1} \circ \cdots \circ \theta_{\beta_k} : H^*(G/T) \to H^{*-2k}(G/T)$$

is well defined (i.e. not depending on the reduced decomposition $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$ chosen). This enables one to evaluate the operator θ_w by using any reduced decomposition of w.

Lemma 5.5. *With respect to the additive basis $\{P_{w'} \mid w' \in W, l(w') = k\}$ of the $2k$-dimensional cohomology $H^{2k}(G/T)$, the operator $\theta_w : H^{2k}(G/T) \to \mathbb{Z}$ is given by $\theta_w(P_{w'}) = (-1)^{l(w)} \delta_{w,w'}$.*

Proof. Let $\varphi_w = w \circ \varphi_{\alpha_1,\cdots,\alpha_k} : S(\alpha; \alpha_1, \cdots, \alpha_k) \to G/T$ be the Bott-Samelson resolution of X_w given in (5.2). As in Example 1 we put

(5.3) $$\beta_1 = \alpha_1, \beta_2 = \sigma_{\alpha_1}(\alpha_2), \cdots, \beta_k = \sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{k-1}}(\alpha_k).$$

Then $\beta_i \in \Delta$, and $w = \sigma_{\beta_1} \circ \cdots \circ \sigma_{\beta_k}$ is a reduced decomposition of w. Consequently,

(5.4) $$\theta_{\beta_i} = \sigma_{\alpha_1}^* \circ \cdots \circ \sigma_{\alpha_{i-1}}^* \circ \theta_{\alpha_i} \circ \sigma_{\alpha_{i-1}}^* \circ \cdots \circ \sigma_{\alpha_1}^*$$

by (5.3) and (5) of Lemma 4.3. From (2) of Lemma 4.4 we have

$$\theta_w = \theta_{\beta_1} \circ \cdots \circ \theta_{\beta_k}$$
$$= (\theta_{\alpha_1} \circ \sigma_{\alpha_1}^*) \circ (\theta_{\alpha_2} \circ \sigma_{\alpha_2}^*) \circ \cdots \circ (\theta_{\alpha_k} \circ \sigma_{\alpha_k}^*) \circ w^* \text{ (by (5.4))}.$$

Substituting in

$$\theta_{\alpha_i} \circ \sigma_{\alpha_i}^* = \theta_{\alpha_i} \circ (Id - e_{\alpha_i}\theta_{\alpha_i}) \text{ (by (3) of Lemma 4.3)}$$
$$= -\theta_{\alpha_i} \text{ (by (1), (4) and (2) of Lemma 4.3)}$$

we get $\theta_w = (-1)^{l(w)}\theta_{\alpha_1} \circ \cdots \circ \theta_{\alpha_k} \circ w^*$. The proof is completed by Lemma 5.4.

5.4. Proof of Lemma 5.1. For an ordered sequence $\{\beta_1, \cdots, \beta_k\}$ of simple roots consider the induced map

$$\varphi_{\beta_1,\cdots,\beta_k}^* : H^{2r}(G/T) \to H^{2r}(S(\alpha; \beta_1, \cdots, \beta_k)).$$

For a $P_w \in H^{2l(w)}(G/T)$ we have

$$\varphi_{\beta_1,\cdots,\beta_k}^*(P_w) = \sum_{|L|=l(w); L\subseteq [1,\cdots,k]} \theta_L(P_w)x_L \text{ (by Lemma 5.2)}$$
$$= \sum_{l(\sigma_L)=|L|=l(w)} \theta_{\sigma_L}(P_w)x_L \text{ (by Lemma 4.4)}$$
$$= (-1)^{l(w)} \sum_{|L|=l(w),\sigma_L=w} x_L \text{ (by Lemma 5.5)}.$$

6 Proof of the Theorem

The Schubert varieties in a generalized flag manifold G/H can be described by those in G/T, where H is the centralizer of a 1-parameter subgroup in G.

Let $\Delta = \{\beta_1, \cdots, \beta_n\}$ be the set of simple roots relative to the regular point $\alpha \in L(T)$ (cf. **2.1**). Assume that $b \in L(T)\backslash\{0\}$ is a point lying in exactly d of the singular hyperplanes $L_{\beta_1}, \cdots, L_{\beta_n}$, say $b \in L_{\beta_1} \cap \cdots \cap L_{\beta_d}$. We set G_b to be the centralizer of the 1-parameter subgroup $\{\exp(tb) \mid t \in \mathbb{R}\}$ in G. It is well known that

(1) *if* $d = 0$, *then* G_b *is the fixed maximal torus* T;

and in general

(2) *every* H *is conjugated in* G *to one of the subgroups* G_b.

By (2) we may assume that H is of the form G_b for some b taking as the above.

Consequently, T is also a maximal torus of H and the Weyl group W' of H, generated by the reflections σ_{β_i}, $k+1 \leq i \leq n$, is a subgroup of W. As in Section 1, we identify the set W/W' of left cosets of W' in W with the subset of W :

$$\overline{W} = \{w \in W \mid I(w') \geq I(w) \text{ for all } w' \in wW'\}.$$

Consider the standard fibration $p : G/T \to G/H$. From [BGG, §5] we have

Lemma 6.1. *If* $w \in \overline{W}$, *the map* p *restricts to a degree* 1 *map* $X_w \to X_w(H)$ *between Schubert varieties; and if* $w \notin \overline{W}$, $p_*[X_w] = 0$.

Consequently, the induced map $p^* : H^*(G/H) \to H^*(G/T)$ *satisfies*

$$p^*[P_w(H)] = P_w,\ w \in \overline{W}.$$

Proof of the Theorem. For a pair $u, v \in \overline{W}$ assume, in $H^*(G/H)$, that

$$P_u(H)\cdot P_v(H)=\sum_{l(w')=l(u)+l(v),w'\in\overline{W}} a_{u,v}^{w'}P_{w'}(H),\ a_{u,v}^{w'}\in\mathbb{Z}.$$

Applying the induced ring map p^* we get the equality

$$P_u\cdot P_v=\sum_{l(w')=l(u)+l(v),w'\in\overline{W}} a_{u,v}^{w'}P_{w'},\ a_{u,v}^{w'}\in\mathbb{Z} \tag{6.1}$$

in $H^*(G/T)$ by Lemma 6.1.

Let $w=\sigma_{\beta_1}\circ\cdots\circ\sigma_{\beta_k}$, $\beta_i\in\Delta$ be a reduced decomposition of an $w\in\overline{W}$, $k=l(u)+l(v)$, and let $A_w=(a_{i,j})_{k\times k}$ be the associated Cartan matrix of w. Consider the Bott-Samelson cycle $\varphi_{\beta_1,\cdots,\beta_k}:S(\alpha;\beta_1,\cdots,\beta_k)\to G/T$ associated to the sequence $\beta_1,\cdots,\beta_k$ of simple roots. Applying the induced ring map $\varphi^*_{\beta_1,\cdots,\beta_k}$ to (6.1) yields in $H^*(S(\alpha;\beta_1,\cdots,\beta_k))$ that

$$\begin{aligned}\varphi^*_{\beta_1,\cdots,\beta_k}[P_u\cdot P_v]&=\sum_{l(w')=l(u)+l(v),w'\in\overline{W}} a_{u,v}^{w'}\varphi^*_{\beta_1,\cdots,\beta_k}[P_{w'}]\\&=(-1)^k a_{u,v}^{w}x_1\cdots x_k,\end{aligned}$$

where the second equality follows from

$$\varphi^*_{\beta_1,\cdots,\beta_k}[P_{w'}]=\left\{\begin{array}{c}(-1)^k x_1\cdots x_k \text{ if } w'=w;\\ 0 \text{ if } w'\neq w\end{array}\right.$$

by Lemma 5.1. On the other hand

$$\varphi^*_{\beta_1,\cdots,\beta_k}[P_u\cdot P_v]=[((-1)^{l(u)}\sum_{\substack{|L|=l(u)\\ \sigma_L=u}}x_L)((-1)^{l(v)}\sum_{\substack{|K|=l(v)\\ \sigma_K=v}}x_K)]$$

(again) by Lemma 5.1. Summarizing we get in $H^{2k}(S(\alpha;\beta_1,\cdots,\beta_k))$ that

$$(-1)^k a_{u,v}^w x_1\cdots x_k=(-1)^{l(u)+l(v)}(\sum_{\substack{|L|=l(u)\\ \sigma_L=u}}x_L)(\sum_{\substack{|K|=l(v)\\ \sigma_K=v}}x_K).$$

Evaluating both sides on the orientation class $[S(\alpha;\beta_1,\cdots,\beta_k)]$ and noting that $< x_1\cdots x_k,[S(\alpha;\beta_1,\cdots,\beta_k)] >=1$ and $k=l(u)+l(v)$, we get by Lemma 3.4 that

$$a_{u,v}^w=T_{A_w}[(\sum_{\substack{|L|=l(u)\\ \sigma_L=u}}x_L)(\sum_{\substack{|K|=l(v)\\ \sigma_K=v}}x_K)].$$

This completes the proof of the Theorem.

7 Historical remarks

7.1. K-cycles in a flag manifold G/T. In 1954 R. Bott constructed a Morse function on G/T and showed that G/T was a cell complex with cells in the even dimensions only. They turn out to be so-called *K-cycles* of Bott-Samelson [BS$_2$] formulated in the following plausible way.

For each root $\beta\in\Phi$ let $K_\beta\subset G$ be the stabilizer of the singular plane $L_\beta\in D(G)$ under the adjoint action of G on $L(G)$. For an ordered sequence $\{\beta_1,\cdots,\beta_k\}$ of roots one forms the products $K(\beta_1,\cdots,\beta_k)=K_{\beta_1}\times\cdots\times K_{\beta_k}$. Since $T\subset K_{\beta_i}$ for each i, the group $T(k)=T\times\cdots\times T$ (k factors) acts on $K(\beta_1,\cdots,\beta_k)$ from the right by

$$(c_1,\cdots,c_k)\cdot(t_1,\cdots,t_k)=(c_1t_1,t_1^{-1}c_2t_2,\cdots,t_{k-1}^{-1}c_kt_k).$$

This defines $K(\beta_1,\cdots,\beta_k)$ as a $T(k)$-principal bundle, whose base manifold is called $K_{\beta_1}\times_T\cdots\times_T K_{\beta_k}$. The point in the base corresponding to $(c_1,\cdots,c_k)\in K(\beta_1,\cdots,\beta_k)$ is denoted by $[c_1,\cdots,c_k]$. The *K-cycle* associated to the sequence $\{\beta_1,\cdots,\beta_k\}$ of roots is the map $f_{\beta_1,\cdots,\beta_k}: K_{\beta_1}\times_T\cdots\times_T K_{\beta_k}\to G/T$ by

$$f_{\beta_1,\cdots,\beta_k}[c_1,\cdots,c_k]=Ad_{c_1\cdots c_k}(\alpha).$$

Certain K-cycles were selected to describe the stable manifolds of a perfect Morse function on G/T, hence provide an explicit additive basis for the homology $H_*(G/T)$ [BS$_2$]. Picture W as the W-orbit $\{w(\alpha)\in L(T)\mid w\in W\}$ of the regular point α. For each $w\in W$ let C_w be a straight line segment in $L(T)$ from the Weyl chamber containing α to $w(\alpha)$ that crosses the planes in $D(G)$ one at a time, and assume that they are met in the order $L_{\alpha_1},\cdots,L_{\alpha_k}$, $\alpha_i\in\Phi^+$. Let $\Gamma_w=K_{\alpha_1}\times_T\cdots\times_T K_{\alpha_k}$ and define $g_w:\Gamma_w\to G/T$ to be the composition $w\circ f_{\alpha_1,\cdots,\alpha_k}$. It was shown in [BS$_2$] that

Proposition 1. *The set of cycles $\{g_{w*}[\Gamma_w]\in H_*(G/T)\mid w\in W\}$ is a basis of $H_*(G/T)$.*

7.2. Bott-Samelson cycles in an isoparametric submanifold. The embedding $\varphi: G/T\to L(G)$ given by the adjoint representation at the beginning of Section 4 defines G/T as an *isoparametric submanifolds* in the Euclidean space $L(G)$ [HPT].

In general, associated to any *isoparametric submanifold* M in an Euclidean space $\mathbb{R}^N$ there are also concepts like (finite) Coxeter group, root system and Dynkin diagram (marked with multiplicities). In order to generalize Bott-Samelson's above cited result to such more general spaces which are also of historical interests in differential geometry, Hsiang-Palais-Terng introduced in [HPT] the space $S(y;\alpha_1,\cdots,\alpha_k)$ as well as the map $\varphi_{\beta_1,\cdots,\beta_k}: S(y;\beta_1,\cdots,\beta_k)\to M$ under the name "*Bott-Samelson cycles*" in the same way as that given in Remark 5, Section 4. The construction of these cycles uses only the integrability of the tangent distributions on M (which are also indexed by sequences of positive roots relative to a non-focal point $\alpha\in\mathbb{R}^N$ of M) *while the groups K_β's required to define K-cycles no long always exist in this more general situation.*

The idea of Bott-Samelson cycles does generalize the K-cycles of Bott-Samelson in the following sense.

Proposition 2 (cf.[Du$_2$, Lemma 8])**.** *If $M=G/T$, there is an orientation preserving diffeomorphism $g: K_{\beta_1}\times_T\cdots\times_T K_{\beta_k}\to S(\alpha;\beta_1,\cdots,\beta_k)$ so that the following mapping triangle commutes*

$$\begin{array}{ccc} K_{\beta_1}\times_T\cdots\times_T K_{\beta_k} & & \\ g\downarrow & \overset{f_{\beta_1,\cdots,\beta_k}}{\searrow} & \\ S(\alpha;\beta_1,\cdots,\beta_k) & \overset{\varphi_{\beta_1,\cdots,\beta_k}}{\to} & G/T. \end{array}$$

7.3. Schubert varieties. Let K be a linear algebraic group over the field $\mathbb{C}$ of complex numbers, and let $B\subset K$ be a Borel subgroup. The homogeneous

variety K/B is a projective variety on which the group K acts by left translations. Historically, Schubert varieties were introduced in term of the orbits of B action on K/B.

Let T be a maximal torus containing in B and let $N(T)$ be the normalizer of T in K. The Weyl group of K (relative to T) is $W = N(T)/T$. For an $w \in W$ take an $n(w) \in N(T)$ such that its residue class mod T is w.

The following result was first discovered by Bruhat for classical Lie groups K in 1954, and proved to be the case for all reductive algebraic linear groups by Chevalley [Ch$_2$] in 1958.

Proposition 3. *One has the disjoint union decomposition*

$$K/B = \underset{w\in W}{\cup} Bn(w) \cdot B$$

in which each orbit $Bn(w) \cdot B$ is isomorphic to an affine space of complex dimension $l(w)$.

The Zariski closure of the open cell $Bn(w) \cdot B$ in K/B, denoted by X_w, is called the *Schubert variety associated to w*.

7.4. Bott-Samelson desingularizations of Schubert varieties. For a compact connected Lie group G with a maximal torus T let K be the complexification of G, and let B be a Borel subgroup in K containing T. It is well known that the natural inclusion $G \to K$ induces an isomorphism $G/T = K/B$. Conversely, the reductive algebraic linear groups are exactly the complexifications of the compact real Lie groups (cf. [Ho]).

It follows now from Proposition 1 and 3 that the homology $H_*(G/T)$ has two canonical additive bases: the first of these is given by the K-cycles of Bott-Samelson; the second consists of Schubert varieties, and both of them are indexed by the Weyl group of G.

Before thinking of the problem on the relationship between these two bases of $H_*(G/T)$ as "*a natural one*", one should bear in mind that the first basis was constructed to provide the stable manifolds of a perfect Morse function on G/T, while the second arose from the jumbled efforts through centuries of great many mathematicians who have contributed to lay the fundation of algebraic intersection theory for the Schubert's enumerative calculus [K$_1$], [K$_2$]. The following result was obtained by Hansen in 1973 [Han].

Proposition 4. *Under the natural isomorphism $G/T = K/B$, the K-cycle $g_w : \Gamma_w \to G/T$ of Bott-Samelson is a degree 1 map onto the Schubert variety X_w.*

Combining Proposition 2 with Proposition 4, we have the following alternative definition of Schubert varieties in G/T (for compact G) without resorting to the complexification of G.

Given an $w \in W$ let C_w be a straight line segment in $L(T)$ from the Weyl chamber containing α to $w(\alpha)$ that crosses the planes in $D(G)$ one at a time. Assume that they are met in the order $L_{\alpha_1}, \cdots, L_{\alpha_k}$, $\alpha_i \in \Phi^+$.

Definition 7. The *Schubert variety* in G/T associated to an $w \in W$ is $X_w = \operatorname{Im} \varphi_w$, where φ_w is the composition

$$\varphi_w = w \circ \varphi_{\alpha_1, \cdots, \alpha_k} : S(\alpha; \alpha_1, \cdots, \alpha_k) \to G/T.$$

This is the version of the descriptions of Schubert varieties that we have made use of in Section 5 and 6.

7.5. Divided differences. The inclusion $T \to G$ of the maximal torus induces a fibration

$$G/T \overset{c}{\hookrightarrow} BT \to BG,$$

where BH denotes the classifying space of a Lie group H , and where the fibre inclusion c is equivariant with respect to the standard W-action on both G/T and BT [B].

Let $H^*(X; \mathbb{R})$ be the cohomology ring (resp. algebra) of a space X with coefficients in the field $\mathbb{R}$ of reals. We recall the classical result due to Borel [B].

Proposition 5. *The map c induces an W-equivariant surjective homomorphism of algebras*

$$c^* : H^*(BT;\mathbb{R}) \to H^*(G/T; \mathbb{R})$$

with kernel $H^+(BT; \mathbb{R})^W$, the ideal in $H^*(BT; \mathbb{R})$ generated by W-invariants in positive degrees.

The induced map c^*, playing a key role in this result, is well known as the *Borel's characteristic map*.

The infinite complex projective space $\mathbb{C}P^\infty$ serves both as the classifying space of the circle group S^1 and the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. Keeping this in mind one get two ingredients from each root $\alpha \in \Phi$.

(1) The reflection σ_α on $L(T)$ in the hyperplane L_α preserves the *unit lattice* $\Lambda = \exp^{-1}(e)$, hence induces in successive manner an automorphism of the torus $T = L(T)/\Lambda$; a diffeomorphism of the classifying space BT, and finally, an induced automorphism σ_α^* of the ring $H^*(BT; \mathbb{R})$.

(2) the *co-root* $\alpha^* : L(T) \to \mathbb{R}$ related to the root $\alpha \in \Phi$ in the fashion $\alpha^*(v) = (\alpha_i, v)$, $v \in L(T)$ satisfies $\alpha^*(\Lambda) \subset \mathbb{Z}$, hence induces successively a homomorphism $T \to S^1$, a map between classifying space $BT \to BS^1 = \mathbb{C}P^\infty$, an finally a 2-cocycle $[\alpha] \in H^2(BT; \mathbb{R})$.

In terms of these, an additive operation $A_\alpha : H^*(BT; \mathbb{R}) \to H^*(BT; \mathbb{R})$ of degree -2 can be defined as follows.

$$A_\alpha(f) = \frac{f - \sigma_\alpha^*(f)}{[\alpha]}, \ f \in H^*(BT; \mathbb{R}).$$

The operator A_α on $H^*(BT; \mathbb{R})$, known as *the divided difference operators associated to the root* α, was introduced independently by Bernstein et al [BGG] and Demazure [De_1] in 1973. These operators possess the same properties as that of the operators θ_α on $H^*(G/T)$ (compare [BGG, 3.3 Lemma] with (2) of Lemma 4.3 in Section 4). In fact, Borel's characteristic map gives rise to a linkage between these two sets of operators.

Proposition 6. *For each $\alpha \in \Phi$, the two operators A_α and θ_α satisfy the commutative diagram*

$$\begin{array}{ccc} H^*(BT;\mathbb{R}) & \overset{c^*}{\to} & H^*(G/T,\mathbb{R}) = H^*(G/T)\otimes\mathbb{R} \\ A_\alpha \downarrow & & \downarrow \theta_\alpha \otimes 1 \\ H^*(BT;\mathbb{R}) & \overset{c^*}{\to} & H^*(G/T,\mathbb{R}) = H^*(G/T)\otimes\mathbb{R}. \end{array}$$

However, by introducing the operators θ_α in the much more general context of spherical represented involutions (cf. **3.2**), and by developing their basic properties (Lemma 3.2) in a way independent of the A_α and the Borel characteristic map, these operators not only can act directly on integral ring $H^*(G/T)$ (in which we are actually interested), but also become applicable in the study of intersection theory of isoparametric submanifolds, transnormal submanifolds, as well as their focal manifolds [HPT], [R]. In the context of Morse theory, these families of manifolds are seen as natural generalizations of flag manifolds in the sense that *generalized Schubert cycles* (cf. [HPT, p.449]) can be constructed for distance functions on these manifolds and which are also indexed by the cosets of certain finite groups. In these general settings there are no analogue of Cartan subalgebra (and consequently, Borel's characteristic maps) available.

7.6. Positively multiplying Schubert classes (continuing from Remark 1, Section 2). Let S be a twisted products of 2-spheres of rank k with structure matrix A, and let $B = \{x_I \mid I \subset [1,\cdots,k]\}$ be the additive basis for $H^*(S)$ specified by Lemma 3.3. Product among the basis elements of complementary dimensions yields the numbers $c_{I,J} \in \mathbb{Z}$ with

$$x_I \cdot x_J = c_{I,J} x_1 \cdots x_k,\ \mid I \mid + \mid J \mid = k.$$

Equivalently, $c_{I,J} = T_A(x_I \cdot x_J)$.

The advantages to work with the basis B are obvious. It consists of the elements Kronecker dual to the ready-made geometric cycles $S(I)$ in S (Lemma 3.3). With respect to this basis the action of Bott-Samelson cycles admits a simple description (Lemma 5.1). As a result the geometric essence of the Theorem is transparent. On the other hand, the product in S between these basis elements are not always positive in the sense that $c_{I,J} < 0$ may occur (cf. Example 3). This explains the reason that our formula falls short of positivity.

The classical Richardson-Littlewood rule for multiplying Schubert classes in the Grassmannian has the merit to meet the standard of positivity. However, it is difficult to summarize the rule into an explicit formula required by effective computation. Its lengthy and technical statement (cf. [St, p.228]) form a sharp contrast with the conciseness of the Theorem.

Concerning the geometric origin of the problem, effective computability should be granted with the first priority among various standards on the multiplicative rules [K_1], [K_2]. Therefore, it is natural to make inquires about a unified solution to the problem in its deserved simplicity and natural generality.

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